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Time-varying (S, s) band models: properties and interpretation*

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Abstract

A recent strand of empirical work uses (S,s) models with time-varying stochastic bands to describe infrequent adjustments of prices and other variables. The present paper examines some properties of this model, which encompasses most micro-founded adjustment rules rationalizing infrequent changes. We illustrate that this model is flexible enough to fit data characterized by infrequent adjustment and variable adjustment size. We show that, to the extent that there is variability in the size of adjustments (e.g. if both small and large price changes are observed), i) a large band parameter is needed to fit the data and ii) the average band of inaction underlying the model may differ strikingly from the typical observed size of adjustment. The paper thus provides a rationalization for a recurrent empirical result: very large estimated values for the parameters measuring the band of inaction.

Keywords: (S,s) models, adjustment costs, menu costs.

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1 Introduction

For many economic behaviors, the decision rule of agents is commonly described by a (S, s) rule. The (S, s) rule model is characterized by the existence of a "band of inaction", i.e. a range of values of the state variable for which it is optimal not to adjust. Such a decision rule can be derived from a dynamic optimization program in the presence of non-convex adjustment costs which are assumed to be relevant in many economic decisions like inventories (Scarf, 1959), prices (Sheshinski and Weiss, 1977) or consumption and investment (Grossman and Laroque, 1990). On the empirical side, (S, s) rules are convenient reduced forms that can be confronted to the data. In particular, they can account for infrequent adjustment observed in microeconomic data, and be tested with such data using standard limited dependent variable methods. First estimations of this class of reduced form models can be found in Sheshinski, Tishler and Weiss (1981) and Dahlby (1992).[1]

However the standard (S, s) model faces some empirical difficulties. Indeed it predicts that, when a firm adjusts, the size of the variation is always equal (or proportional) to the size of the band of inaction. In particular, all adjustments are of similar magnitude and are rather large. This prediction is at variance with patterns often observed in microeconomic data. For example, Hall and Rust (2000) report a high degree of variability in investment adjustment decisions, which is highly difficult to match with a fixed band of inaction model. In the case of prices, the prevalence of small price changes and the significant variance in the size of microeconomic price adjustments has been widely documented, see inter alia Dhyne *et al.* (2006) and Klenow and Kryvstov (2008).

[1] See also Foote, Hurst and Leahy (2000) and the references therein.

To overcome this difficulty, a growing strand of research has proposed empirical models allowing for time-varying random (S, s) bands. Caballero, Engel and Haltiwanger (1997) use a model with time-varying bands to rationalize microeconomic employment adjustment policies that vary over time. Recently, Fisher and Konieczny (1995, 2006) and Dhyne, Fuss, Pesaran and Sevestre (2007) estimate (S, s) models with random thresholds using individual price data for several categories of products. As discussed by Caballero and Engel (1999) and Hall and Rust (2000), such time-varying random (S, s) bands can be rationalized by models assuming a random adjustment cost. Stochastic adjustment costs then give rise to sizes of adjustment that vary over time for a given firm.

This paper investigates some properties of reduced-form random (S, s) band models, and provides some guidelines to interpret the results obtained using that class of models. Several of our results are related to properties obtained in structural models with random menu costs by Willis (2000) and Dotsey, King and Wolman (2009). Here, focusing on a reduced form allows us to provide some analytical results and thus make the underlying mechanisms more explicit. This focus allows a closer connection to microeconometric models and estimations.

Our contribution is twofold. First, we exhibit some relationships between the mean and variance of the (S, s) band and different moments of adjustments generated by the model. We are able to derive analytically these relationships using a simple framework. Formally, the problem amounts to studying the properties of the waiting time before hitting a random barrier. Though our results relate to a simple framework, to our knowledge, they have not been previously established in the literature. Further, using simulations, we illustrate that similar properties hold in more elaborate models.

Second, we rationalize a recurrent finding in the empirical literature: the rather large size of the estimated mean bandwidth, typically much larger than the average observed price change. For example, Attanasio (2000) estimates time-varying (S, s) band model on durable goods data, and reports that the average size of the band is wider than the average size of price changes.[2] Dhyne *et al.* (2007) also underline this result, which stands in contrast with the deterministic version of the model where the size of price changes is expected to be equal to the size of the inaction band. We explain this result by a simple property of the model: introducing variability in the adjustment threshold increases the variance of adjustment sizes but also reduces the average size of adjustment. Since the average size of adjustment is itself an increasing function of the bandwidth, it turns out that fitting data with substantial variability in size of adjustment requires both the mean and the variance of the (S, s) band to have large values.

This paper is structured as follows. Section 2 provides an economic motivation by considering a structural menu cost model with random adjustment costs, which gives rise to random (S, s) bands. Within a simple framework, Section 3 establishes some analytical results on the relationship between the variance and the mean of the band and moments generated by the model (hazard rate, mean and variance of adjustments). Using more realistic models, Section 4 provides simulation evidence which confirms analytical results. In Section 5, we illustrate our results using actual estimates from the literature. Section 6 summarizes and draws implications for interpreting empirical evidence.

[2] Attanasio (2000) notes that "the most striking feature, however, is the width of the band".

2 Time-varying (S,s) band models: a structural motivation

In this section we provide a structural motivation to the random (S,s) band model. We show that this model is able to describe the optimal microeconomic policy rule when menu costs are stochastic. In the case of price adjustment rules, Sheshinski and Weiss (1977) show that in the presence of a fixed menu cost and constant inflation, (S,s) policies are optimal. As obtained by Caballero and Engel (1999) and Hall and Rust (2007), if menu costs are random, the optimal policy rules can be represented by models with stochastic (S,s) bands. To our knowledge, the optimality of these generalized forms of the (S,s) policy has not been proved analytically in a general case (see Caballero and Engel (1999) for a discussion). Here, in line with earlier literature, we use simulations of a calibrated model with stochastic menu costs, solved numerically. We consider price adjustment rules and rely on a menu cost model comparable to the models recently analyzed by Dotsey, King and Wolman (1999), Golosov and Lucas (2008), Klenow and Krystov (2008), and Nakamura and Steinsson (2008).

More precisely, we use Nakamura and Steinsson's (2008) set-up, and extend their model by introducing a stochastic rather than deterministic menu cost. This model considers, in a partial equilibrium context, the pricing decision of a firm that operates in a monopolistic competition environment. The demand adressed to the firm is $Y_t^d = D\left(\frac{P_t}{\overline{P}_t}\right)^{-\theta}$, where P_t is the firm's price, $\overline{P}_t$ is the overall price level and D is constant which corresponds to the steady-state output level, we set it equal to 1 here. The production function of the firm is linear: $Y_t = A_t N_t$ where A_t is the level of productivity and N_t total hours worked. The logarithm of productivity is assumed to follow an $AR(1)$ process: $a_t = \rho a_{t-1} + \varepsilon_t$ where $a_t = ln(A_t)$ and $\sigma_\varepsilon^2 = E\varepsilon_t^2$. The overall price level is assumed to follow a random walk with drift $ln\overline{P}_t = \mu^P + \ln(\overline{P}_{t-1}) + \varepsilon_t^P$. As Nakamura and Steinsson (2008), we assume that the real wage is constant and equal to its equilibrium level under flexible price, $\frac{W_t}{\overline{P}_t} = \frac{\theta-1}{\theta}$. The period real profit function is then given by:

$$\Pi(P_t/\overline{P}_t, a_t) = \frac{P_t}{\overline{P}_t} D \left(\frac{P_t}{\overline{P}_t}\right)^{-\theta} - \left(\frac{\theta-1}{\theta}\right) \frac{Y_t}{\exp(a_t)}$$

When changing its price, the firm incurs a menu cost $c_t D$, where c_t is the menu cost expressed as a fraction of steady state output D. We here assume that c_t is stochastic, and drawn from a Beta distribution (following Dotsey, King and Wolman (1999)). This specification generates positive and bounded menu costs, but still allows for a wide range of cases, including as specific cases: a fixed menu cost and a bimodal distribution, which mimicks the Calvo process. The vector of state variables is $\{P_{t-1}/\overline{P}_t, a_t, c_t\}$. At time t, assuming a discount factor of β,the value of the firm (the present value of profits) is given by:

$$V(P_{t-1}/\overline{P}_t, a_t, c_t) = \max[V^{nc}(P_{t-1}/\overline{P}_t, a_t, c_t), V^c(P_{t-1}/\overline{P}_t, a_t, c_t)]$$

where $V^c(P_{t-1}/\overline{P}_t, a_t, c_t)$ is the value if the firm change its prices and $V^{nc}(P_{t-1}/\overline{P}_t, a_t, c_t)$ is the value if the firm does not change its price. These two functions are given by:

$$V^c(P_{t-1}/\overline{P}_t, a_t, c_t) = \max_{p_t}[\Pi(P_t/\overline{P}_t, a_t) + \beta E_t V(P_t/\overline{P}_{t+1}, a_{t+1}, c_{t+1})] - c_t D$$

and

$$V^{nc}(P_{t-1}/\overline{P}_t, a_t, c_t) = \Pi(P_{t-1}/\overline{P}_t, a_t) + \beta E_t V(P_t/\overline{P}_{t+1}, a_{t+1}, c_{t+1})$$

To solve the model, we use value-function iteration. For this purpose the processes for productivity, inflation and menu costs are discretized. The Tauchen (1986) procedure is used to discretize the processes for productivity and inflation, while discretization of c_t is straightforward given independence across draws. We then solve the program of the firm and are able to derive the policy function and simulate the model.

We calibrate the model following Nakamura and Steinsson (2008)'s results on US monthly price data. We set the discount factor to $\beta = 0.96^{1/12}$, and the elasticity of demand to $\theta = 4$. Both values fall in standard ranges, and the latter is consistent with a mark-up of 1.33. Concerning the productivity shocks, we set $\sigma_\varepsilon = 0.0428$ and $\rho = 0.66$ based on Nakamura and Steinsson (2008)'s results. The mean of the process for overall inflation is set to $\mu^P = 0.0021$ and the standard deviation of ε_t^P to $\sigma_\varepsilon^P = 0.0032$ from the data on US CPI. The menu costs are assumed to be independently drawn from a Beta distribution with mean $\mu^c = 0.04$, and standard deviation $\sigma_c = 0.03$.[3] Note that these moments characterize the population distribution of menu costs, but may not characterize the empirical distribution of menu costs actually paid by firms since firms will not change price independently of the realized value of the menu cost (see Section 5). Nakamura and Steinsson (2008) assume a fixed menu cost equal to 0.0245.

To represent the pricing policy of the firm and relate this model to empirical (S, s) models, a relevant variable is the price gap, introduced in particular by Caballero and Engel (1999). If prices were flexible, the nominal optimal price would be $P_t^* = (\frac{\theta}{\theta-1})\frac{W_t}{A_t}$. Hence under the above assumption the log-optimal price is $p_t^* = \overline{p}_t - a_t$. We define the price gap of a firm at date t as $z_t = p_{t-\tau} - p_t^*$, where τ is the duration elapsed since the last price change. The decision rule can then be expressed as a function of z_t and c_t following Caballero and Engel's (1999) approach.

[3] The corresponding parameters (a, b) of the standard parametrization of the Beta distribution are $a = 1.7$ and $b = 40.0$.

The decision rule is pictured in Figure 1.[4] The realization of the menu cost is on the x-axis and the level of the pre-adjustment price gap z_t is on the y-axis. For each value of menu cost, the solid line gives the threshold values for which the firm is indifferent between changing in price and keep its price unchanged. Inside the region drawn by the curve, the price is kept unchanged whereas outside this region, the price is changed. The dotted line represents the reset point. The reset point is observed to be larger than zero: this reflects that the drift in inflation mechanically erodes real price, so the firm paying the menu cost when changing its price anticipates this erosion and sets its price above the frictionless optimal price.

[Figure 1 here]

The main point we want to emphasize here is that this solid line describes the (S, s) band obtained for each realization of the menu cost. Figure 1 illustrates that the inaction band is varying with the value of the menu cost: for larger realizations of menu costs, the band is larger.[5] This result is in accordance with the inaction band obtained by Caballero and Engel (1999) for investment. This exercise provides a structural motivation to the empirical model we study hereafter.

In the following exercises, we will assume that variations in (S, s) bands are only due to the variations of the menu costs but other shocks in the model could generate time-variation in (S, s) bands. For instance Golosov and Lucas (2008) show that productivity shocks can lead to

[4] For simplicity we plot the figure only for a particular value of a_t.

[5] It can be noted that there is an asymmetry in the inaction zone, for the same reason for which the reset price is above zero.

varying adjustment bands. The boundaries of the inaction zone in Figure 1 are also dependent on parameters μ^P and σ_ε^P.

3 Properties of a time-varying (S, s) band models: some analytical results

In this section, using a simple model, we derive analytically some properties of the reduced-form (S, s) model with time-varying thresholds. In particular, we show how the data moments generated by the model are related to the parameters of the model.[6] We focus here on a price-setting decision rule but our results can obviously be extended to other types of economic decisions.

3.1 A simple model

We note respectively p_t the logarithm of the price posted by the firm at date t and p_t^* is the logarithm of the reset price, i.e. the price it chooses to implement if it changes prices at date t.[7] For simplicity, we consider a model that only involves price increases. We assume that the policy of the firm is to follow a one-sided (S, s) rule, and that the gap $p_t^* - p_t$ fully describes the environment of the agent. We note S_t the time-varying threshold for price increases. That is,

[6] We underline that we map the moments of the data with the parameters of the reduced form model, not the "deep" parameters like the variance and the mean value of the menu cost. Assessing whether the latter mapping is analytically tractable is left for future research.

[7] The underlying structural models (e.g. Sheshinski and Weiss, 1977) typically predict that, under positive inflation, the reset price is equal to the optimal frictionless price plus a positive constant. For simplicity, we disregard this constant.

the firm's policy is to maintain its price unchanged as long as $p_t^* - p_{t-\tau} < S_t$ and to change its price to p_t^* whenever $p_t^* - p_{t-\tau} \geqslant S_t$.

To obtain analytical results, we make the following assumptions on the processes for p_t^* and S_t in this section. First, the reset price follows a deterministic trend $p_t^* = \gamma t$ where $\gamma > 0$. Without restriction, we set the initial nominal price to be $p_0 = 0$ and the initial reset price to be $p_0^* = 0$. Second, the price threshold follows a Bernoulli distribution: $S_t = S + \nu_t$ where $P(\nu_t = a) = P(\nu_t = -a) = \frac{1}{2}$. Thus, at each date the firm can either face, with an equal probability, a "low" threshold or a "high" threshold.

The model is summarized in Figure 2: the price deviation p_t^* grows until it hits the random threshold S_t. Observe that here the price can only be modified between dates $t = T^-$ and $t = T^+$ characterized by $p_{T^-}^* = \gamma T^- = S - a$ and $p_{T^+}^* = \gamma T^+ = S + a$. Thus $T^- = \frac{S-a}{\gamma}$ and $T^+ = \frac{S+a}{\gamma}$. For convenience we assume that $\frac{S-a}{\gamma}$ and $\frac{S+a}{\gamma}$ are integer numbers, which also implies that $T^+ - T^- = \frac{2a}{\gamma}$ is a strictly positive integer (note that $\frac{2a}{\gamma} \geq 1$, so that $a \geq \frac{\gamma}{2}$).

[Figure 2 here]

3.2 Adjustment hazard

In this paper, we characterize the distribution of price changes. In our simple set-up, the size of price changes is γt if price changes at date t, so the size of price changes is an obvious function of the duration of the first price spell. We note τ this random variable, which is the waiting time before hitting the threshold. We then use the hazard function approach to derive the distribution of τ. This approach is insightful in the present context. The probability of price change after t periods can be written as the product of the probability of observing no price change for $t-1$

periods and the conditional probability of price change after t periods. The former term is the survival function $R(t)$ and the latter one is the hazard function $h(t)$. Formally,

$$
\begin{aligned}
P(\Delta p_t &\neq 0, p_{t-1} = ... = p_0) \\
&= P(p_{t-1} = ... = p_0) \times P(\Delta p_t \neq 0 | p_{t-1} = ... = p_0) \\
&= R(t) \times h(t)
\end{aligned}
$$

As observed before, the price can only be modified between dates T^- and T^+. So, if $t \leqslant T^-$ then $R(t) = 1$ and $t > T^+$ then $s(t) = 0$. If $t \in [T^-; T^+]$, the survival function is the probability that the "high threshold" has been realized for $t - T^-$ periods, that is:

$$
\begin{aligned}
R(t) &= P(p_{t-1} = ... = p_{T^-}) \\
&= \left(\frac{1}{2}\right)^{t-T^-}
\end{aligned}
$$

The hazard function is:

$$
\begin{aligned}
h(t) &= P(\Delta p_t \neq 0 | p_{t-1} = ... = p_0) \\
&= P(p_t^* - p_0 \geq S_t) \\
&= P\left(\gamma t \geq S + \nu_t\right)
\end{aligned}
$$

Using $T^- = \frac{S-a}{\gamma}$,

$$
\begin{aligned}
h(t) &= P\left(\gamma t \geq \gamma T^- + a + \nu_t\right) \\
&= P\left(\nu_t \leqslant -a + \gamma(t - T^-)\right)
\end{aligned}
$$

Given the process for ν_t, the hazard function can only take three values in our model $\{0, \frac{1}{2}, 1\}$. For $t < T^-$, $h(t) = 0$. For $T^- \leq t < T^+$, $h(t) = \frac{1}{2}$. In that case indeed, at each period, given that the price has not be changed for $(t-1)$ periods, the probability of price change is $\frac{1}{2}$ (the probability of hitting the lower the band). Lastly, if $t = T^+$, $h(t) = 1$ since $-a+\gamma(t-T^-) = a$, so that $P(\nu_t \leqslant -a + \gamma(t - T^-)) = 1$. In this case, the current price deviation hits the upper limit of the stochastic band, and the probability of price change is equal to one.

To summarize, the hazard function is the following stepwise function:

$$\begin{aligned}
h(t) &= 0 && \text{if } t < \frac{S-a}{\gamma} \\
h(t) &= \frac{1}{2} && \text{if } \frac{S-a}{\gamma} \leq t < \frac{S+a}{\gamma} \\
h(t) &= 1 && \text{if } t = \frac{S+a}{\gamma}
\end{aligned}$$

We can obtain the distribution of the waiting time τ as the product of $h(t)$ and $R(t)$.

$$\begin{aligned}
P(\tau &= t) = 0 && \text{if} \quad t < \frac{S-a}{\gamma} \\
P(\tau &= t) = \left(\frac{1}{2}\right)^{t-\frac{S-a}{\gamma}+1} && \text{if} \quad \frac{S-a}{\gamma} \leq t < \frac{S+a}{\gamma} \\
P(\tau &= t) = \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} && \text{if} \quad t = \frac{S+a}{\gamma}
\end{aligned}$$

It will be useful to consider the distribution of waiting time once date T^- is elapsed, that is to consider the random variable $\tilde{\tau} = \tau - T^- + 1$. The distribution of $\tilde{\tau}$ is as follows:

$$\begin{aligned}
P(\tilde{\tau} &= k) = 0 && \text{if} \quad k < 1 \\
P(\tilde{\tau} &= k) = \left(\frac{1}{2}\right)^{k} && \text{if} \quad 1 \leqslant k < \frac{2a}{\gamma} + 1 \\
P(\tilde{\tau} &= k) = \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} && \text{if} \quad k = \frac{2a}{\gamma} + 1
\end{aligned}$$

Note this distribution is a geometric distribution with parameter $\frac{1}{2}$ and truncated at value $\frac{2a}{\gamma}+1$.

Before proceeding, we can relate the above result to the notion of adjustment hazard that has been introduced by Caballero and Engel (1993 and 1999) to analyze generalized (S, s) models. In such models, the probability of a price change depends on the gap z_t between the optimal price at date t and the current price at that date, i.e. $z_t = p_t^* - p_0$. The probability of a price change expressed as a function of gap variable is called the adjustment hazard function. In our simple framework here $z_t = \gamma t$, so the adjustment hazard is simply $\Lambda(z_t) = h\left(\frac{z_t}{\gamma}\right)$. From above, we see that the adjustment hazard is non-decreasing here. This matches the "increasing hazard" property: Caballero and Engel (1993 and 1999) have shown that the adjustment hazard increases with z_t in generalized (S, s) models.

We can also describe how the adjustment hazard varies with the average threshold S and the band variability a. The hazard is a non-increasing function of S. For large values of S, the range of price gaps z_t's for which the hazard function is null is wider, whereas the range of values of z_t for which the hazard function is 0.5 or 1 is moved to the right. Variations in a, the standard deviation of S_t, have more complicated effects on the hazard. For larger values of a, the range of z_t's for which the hazard function is equal to 0.5 broadens whereas the range of z_t for which the hazard function is 0 is narrower. At the same time, the set of z_t's for which the hazard function is equal to 1 is moved to the right. An extreme case appears when a is very large, implying large variations in S_t. In that case, the adjustment hazard is constant (equal to $\frac{1}{2}$) and the probability of price change does not depend on z_t, as in the Calvo (1983) model. Another extreme case is $a = 0$, $S_t = S$: $\Lambda(z_t)$ is then equal to 1 when $z_t = S$ and 0 otherwise. This case corresponds to the standard (S, s) model with time-invariant bands. Overall, we observe that increasing a flattens the adjustment hazard function.

3.3 Moments of price changes

In this section, we compute the first and second moments of the size of price changes, conditional on a price change being observed. These moments are functions of parameters, which we denote $m_1(S, a, \gamma)$, $m_2(S, a, \gamma)$. Our purpose is to derive qualitative properties on the way parameters S and a can match observed data. While dependence of the moments on γ is acknowledged, we do not focus on the implication of alternative values of γ since empirically, γ is typically not a free parameter. The frictionless price or its determinants are usually observed, so γ can be pinned down from an auxiliary regression.

Here, we only investigate moments of order 1 and 2 since the focus of our paper is on interpreting empirical estimations of time varying (S, s) band models, which typically emphasize the mean and variance of the bandwidth. Investigating higher order moments would be an interesting avenue for future research since high kurtosis has recently been outlined to be a key characteristics of the price change distribution (Midrigan, 2009).[8]

3.3.1 Average price change

We first compute the mean of price changes when a price change is implemented. The first moment of price changes is a function of parameters, which we denote $m_1(S, a, \gamma)$. Given the

[8] Unreported simulation evidence from our model in section 4 suggests time varying random (S, s) bands can actually produce large kurtosis, even if the underlying processes for optimal price and threshold are kept gaussian.

distribution of the price duration τ obtained in the previous section, we have:

$$\begin{aligned}
m_1(S,a,\gamma) &= E(\Delta p_\tau | p_\tau \neq p_{\tau-1}, p_{\tau-1} = ... = p_0) \\
&= E(\gamma\tau) \\
&= E\left(\gamma\left(\widetilde{\tau} + T^- - 1\right)\right) \\
&= E\left(\gamma\left(\frac{S-a}{\gamma} - 1\right) + \gamma\widetilde{\tau}\right) \\
&= S - a - \gamma + \gamma\left[\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma} + 1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right]
\end{aligned}$$

After some algebra we obtain an analytical expression appearing in Proposition 1.

Proposition 1

$$m_1(S,a,\gamma) = S - a + \gamma\left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)$$

Dem: see Appendix 1

From proposition 1, we derive three properties:

Properties

(1.1)

$$\frac{\partial m_1(S,a,\gamma)}{\partial S} = 1 > 0$$

(1.2) Under assumption that $a > \frac{\gamma}{2}$,[9]

$$m_1(S,a,\gamma) < S$$

(1.3)

$$\frac{\partial m_1(S,a,\gamma)}{\partial a} < 0$$

[9] Assumption $a \geq \frac{\gamma}{2}$ is discussed in section 3.1. For the specific case $a = \frac{\gamma}{2}$, see appendix 1.

Property (1.1) is straightforward. Properties (1.2) and (1.3) are demonstrated in the Appendix 1.

Properties (1.1) and (1.2) relate the average size of price changes to S. As expected, the average price change increases with the value of the threshold. Also, Property 1.2 indicates that the average size of price increases is lower when there is time variation in the band ($a > 0$) than when the band is deterministic ($a = 0$).

Property (1.3) characterizes the relation between the average size of price changes and the standard deviation of the band. Recall that a is an index of the variability of the threshold since the standard deviation of S_t is equal to $\sigma_S = a$. The average size of price changes decreases with the standard deviation of the band. This result is crucial in our context. In particular, allowing for a large variance of the band, the model can generate a very small (here arbitrarily small) average size of observed price changes.

3.3.2 Variance of price changes

We now compute the variance of the size of price changes, conditional on a price change being implemented:

$$\begin{aligned}
m_2(S, a, \gamma) &= V(\Delta p_\mathcal{T} | p_\mathcal{T} \neq p_{\mathcal{T}-1}, p_{\mathcal{T}-1} = ... = p_0) \\
&= V(\gamma\mathcal{T}) \\
&= \gamma^2 V\left(\left(\frac{S-a}{\gamma} - 1\right) + \widetilde{\mathcal{T}}\right) = \gamma^2 V(\widetilde{\mathcal{T}}) \\
&= \gamma^2 \left[\sum_{k=1}^{\frac{2a}{\gamma}} k^2 \left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma} + 1\right)^2 \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right]
\end{aligned}$$

After some algebra we obtain an analytical expression appearing in Proposition 2.

Proposition 2

$$m_2(S,a,\gamma) = \gamma^2 \left[2 - \left(\frac{4a}{\gamma} + 1 \right) \left(\frac{1}{2} \right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2} \right)^{\frac{4a}{\gamma}} \right]$$

Dem: see Appendix 1

From proposition 2, we derive two properties:

Properties

(2.1)

$$\frac{\partial m_2(S,a,\gamma)}{\partial S} = 0$$

(2.2)

$$\frac{\partial m_2(S,a,\gamma)}{\partial a} > 0$$

Property (2.1) is straightforward. Property (2.2) is demonstrated in the Appendix 1.

Property (2.1) relates the variance of price changes to S. We find that the variance of price changes is invariant to the value of the threshold S.

Property (2.2) characterizes the relation between the variance of price changes and the standard deviation of the band. We find that the variance of price changes increases with the standard deviation of the band.

To illustrate how these properties translate into actual data, let us consider the case of a model with parameters (S, a) which generates data with mean and variance m_1 and m_2.[10] Now, assume we observe other data characterized by the same mean $m_1^* = m_1$ but a higher variance $m_2^* > m_2$, those data are assumed to be generated by a similar model and a parameter set (S^*, a^*). Given that m_2 does not depend on S, Property (2.2) indicates that the variance of the

[10] As stated above, we assume that γ is typically not a free parameter

threshold is necessarily larger in the second case: $a^* > a$. Now, given Properties (1.1) and (1.3), observing the same data mean $m_1^* = m_1$ with a large band variability $a^* > a$ is only possible if $S^* > S$. A larger value of S^* balances the fact that m_1 decreases with a. This is one of the main findings of the present paper: to fit data with a substantial variance in adjustment size (and a given mean) both a large bandwidth and a large variance of the threshold are necessary.

4 Properties of a time-varying (S, s) band models: simulation evidence

In this section, we use simulations to illustrate how the results presented above extend to other specifications of the threshold S_t that, to our knowledge, to do not lead to analytically tractable problems.

4.1 Simulation design

We focus on the case where the threshold follows a Gaussian distribution. This assumption brings us close to specifications used in the empirical studies (see for example, Fisher and Konieczny (2006) or Dhyne *et al.* (2007)). We build a simulation exercise where the frictionless optimal price is defined as: $p_t^* = \gamma t$ where $\gamma > 0$ and $p_0^* = 0$ and the price threshold is defined as: $S_t = S + \nu_t$ where $\nu_t \sim N(0, \sigma_\nu^2)$.[11] We define τ as the elapsed duration since last price change. The price is changed according the following rule. If $p_t^* - p_{t-\tau} \geq S_t$ (with $\tau > 1$), the observed nominal price is changed and the new price is set to $p_t = p_t^* = p_{t-\tau} + \gamma\tau$. If $p_t^* - p_{t-\tau} < S_t$,

[11] Findings are robust to considering stochastic processes for p_t^*, as detailed below and in Appendix 2.

then the price remains unchanged, i.e. $p_t = p_{t-\tau}$. At each date of price change $\{t|p_t \neq p_{t-1}\}$, we compute the size of the price change. We then compute the average size of non-zero price changes m_1 and the variance of price changes m_2 by simulating the process for a very large sample. This exercise is repeated for different values of S and σ_ν. Results presented in this section have been obtained with samples of size $T = 3,000,000$, and parameter values $p_0^* = 0$, $\gamma = 0.25$, $S \in [0.5; 25.0]$ and $\sigma_S \in [0.5; 10.0]$. Those ranges of parameters produce a wide range of values for the mean and variance of price changes that encompass values reported in the existing empirical literature on micro price adjustments.

4.2 Adjustment hazard function

In this framework, the adjustment hazard function can be written as:

$$
\begin{aligned}
h(z_t) &= P(\Delta p_t \neq 0 | p_{t-1} = ... = p_{t-\tau}) \\
&= P(p_t^* - p_{t-\tau} > S_t) \\
&= P\left(z_t \geq S + \nu_t\right) \\
&= P\left(\gamma\tau - S \geq \nu_t\right) \\
&= \Phi\left(\frac{\gamma\tau - S}{\sigma_S}\right)
\end{aligned}
$$

where Φ is the cumulative distribution function of a standard normal distribution.

Then, three results appear:

(1) $\frac{\partial h(z_t)}{\partial S} < 0$. The hazard decreases with the average band, which is consistent with the findings of section 2.2.

(2) if $z_t - S < 0$ then $\frac{\partial h}{\partial \sigma_S} > 0$ and if $z_t - S > 0$ then $\frac{\partial h}{\partial \sigma_S} < 0$. Like in the discrete case, an

increase in σ_S flattens the hazard adjustment function.

(3) Two extreme cases can be exhibited. If both σ_S and S are large (say, $\sigma_S = \kappa S$ for a positive κ and S is arbitrarily large), we obtain that $h(z_t)$ is close to a constant. The probability of price change does not depend on z_t any more and the hazard adjustment function is flat. This matches the hazard function generated by a Calvo model. The second extreme case is $\sigma_S = 0$. This is the deterministic band case: the hazard is equal to 1 when $z_t = S$ and 0 otherwise.

To illustrate these properties, we plot on Figure 3 the adjustment hazard functions obtained for different values of S and σ_S. We first notice that adjustment hazards are increasing functions of z_t (on x-axis). Thus, the probability of price change increases with the disequilibrium between the observed nominal price and the optimal price. The hazard functions are thus in accordance with hazard functions generated by structural models with stochastic menu costs: see Caballero and Engel (1993 and 1999), Willis (2000) (in the case of price-setting) and section 2 of the present paper.

[Figure 3 here]

Secondly, adjustment hazard functions are decreasing with S (compare the case $S = 10$ in bold line with the case $S = 16$ in thin line). If S is large, as in the case $S = 16$ the range of z_t's for which the hazard is zero is wide and very few price changes are then observed.

Third, the adjustment hazard function flattens with σ_S. In the case $S = 10$, we can distinguish two regions: for z_t between 0 and 10, increasing σ_S from $\sigma_S = 2$ (solid line) to $\sigma_S = 4$ (dashed line) increases the hazard whereas for z_t higher than 10, increasing σ_S leads to a decrease of the hazard.

Finally, we plot an extreme case (in dotted line) where $S = 40$ and $\sigma_S = 80$. This case illustrates our observation (3) above. The hazard function is close to flat and its value does not depend on z_t any more as in a Calvo model. In that case, the hazard function is non-zero even for very small values of z_t, i.e. small price changes are likely to be observed.

Overall, these results are quite consistent with those obtained with a more structural model in which menu costs are randomly distributed (Caballero and Engel (1999) or our model in section 2). In particular the simple DGP used here is able to generate various patterns of the adjustment hazard that all fulfill the increasing adjustment hazard property and are consistent with a structural random menu cost model.

4.3 Moments of price changes

In Figure 4, we report how the average size of price changes m_1 depends on the model parameters S (x-axis) and σ_S (y axis). We can first observe that the average size of price changes is increasing with S, as in Property (1.1) obtained above with a Bernoulli process for S_t. In addition, Figure 4 also illustrates Property (1.3): for a given bandsize parameter, increasing the variance of the band σ_ν decreases the average size of observed price changes. This result appears more clearly on Figure 5 which is a slice of Figure 4: we set $S = 5$ and plot the relationship between m_1 (on y-axis) and σ_ν (on x-axis). The average size of price changes is a decreasing function of the variance of the stochastic band: when the band varies a lot, price adjustments tend to be smaller on average. Lastly, Figures 4 and 5 indicate that the average price change is always smaller than the mean size of the band S, reflecting Property (1.2).

[Figure 4 here]

[Figure 5 here]

Figure 6 plots the variance of price changes as a function of model parameters S (x-axis) and σ_S (y-axis). Results here are partly different from those obtained in the case of a discrete process for S_t. In the region containing large values of S and the low values of σ_S, results are consistent with Properties (2.1) and (2.2): the variance of price changes is insensitive to the variation of S whereas increase in σ_S leads to larger values of m_2. However, in the region containing small values of S and high values of σ_S, we find that increasing σ_S decreases m_2 and also that m_2 is an increasing function of S. The intuition is that there is a maximum for the variance of price changes that can be produced with the model. Indeed, when the variance of the band is large, price changes will tend to be smaller, due to the mechanism of Property (1.3). Since all price changes are clustered in the zone of small price changes, the variance of price changes can not be arbitrarily large. In that region, increasing S relaxes the constraint on the variance of price changes.

[Figure 6 here]

Figure 7 illustrates Properties (1.1), (1.3), (2.1) and (2.2), by representing the contour lines associated with Figures 4 and 6 on the same graph. S and σ_S are represented on the x-axis and y-axis respectively. Contour lines for m_2 are the "L-shaped" curves, with values closest to the North-East corresponding to largest values of m_2. Contour lines for m_1 are the nearly straight lines. Two contour lines are drawn $m_1 = 6$ and $m_1 = 10$. The two regions described above for m_2 are visible from the graph. In the upper-left part of the graph, increasing σ_S leads to lower values of m_2 and increasing S leads to increase m_2. This region is characterized by lower values of m_1. In the lower-right region of the graph (say below contour line $m_1 = 6$, a zone where price changes are larger on average) m_2 does not respond to changes in S whereas it increases with σ_S. Overall, we observe in both regions that to attain a larger variance of price changes (say from $m_2 = 1$ to $m_2 = 2$) while maintaining the same average price change (for example $m_1 = 6$), the model requires both larger values of σ_S and S.[12]

[Figure 7 here]

To further illustrate this last result, we perform a moment matching exercise. We set $m_1 = 6$. and we consider values of m_2 ranging 1 and 3. We then identify the underlying parameters σ_S and S through a minimum distance procedure. Namely for each candidate value (S, σ_S), simulated moments $\widetilde{m}_1(S, \sigma_S)$ and $\widetilde{m}_2(S, \sigma_S)$ are computed using the same simulation exercise as described above (with a trajectory of size $T = 3 \times 10^6$) and a numerical optimization routine is used to set the distance $(m_1 - \widetilde{m}_1(S, \sigma_S))^2 + (m_1 - \widetilde{m}_1(S, \sigma_S))^2$ to zero. Figure 8 plots values of S obtained by this simulated method of moments, as a function of the target moment m_2 (with $m_1 = 6$). As we noticed before, given m_1, matching a larger values of m_2 requires larger values of both σ_S and S.

[Figure 8 here]

Our results are here obtained with specific processes for p_t^* and S_t. However, as mentioned above, our results are also robust to considering more general processes. In Appendix 2, we

[12] Even for very large values of S and σ_S, for given values of m_1 and m_2, contour lines never cross each other more than once (see Figure E in Appendix 2).

provide some results obtained for a different process for p_t^*, we have now $p_t^* = \gamma t + \varepsilon_t$ where $\gamma > 0$ and $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ (we perform different exercises with $\sigma_\varepsilon = 1$, $\sigma_\varepsilon = 2$ and $\sigma_\varepsilon = 4$), the rest of the simulation exercise remains the same. Note that this data-generating process allows for price decreases. All the results obtained with $\sigma_\varepsilon \neq 0$ turn out to be qualitatively in line with those of the baseline case $\sigma_\varepsilon = 0$. It is nevertheless obvious that for very large values of σ_ε, the influence of σ_S on the results becomes weaker. Overall, though further analytical investigation of those issues is left for future research, we conjecture that Properties (1.1) to (2.2) are valid for a very wide range of processes.

5 Interpreting actual estimates

In this last section, we use actual estimation results from Dhyne, Fuss, Pesaran and Sevestre (2007) to illustrate the properties investigated above. Dhyne *et al.* (2007) estimate an (S, s) model with stochastic bands using individual price quotes of more than two hundred products (sold in Belgium and France). Their model is very close to the one presented in Section 3. For each product, the frictionless optimal price in their framework is defined as $p_t^* = f_t + \varepsilon_t$ where f_t is a common factor representing a common component to all outlets and ε_t is an idiosyncratic shock defined as $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. The price change rule is the following: a price is modified as soon as $|p_t^* - p_\tau| > S_t$ where $S_t = S + \nu_t$ with τ the date of the last price change and $\nu_t \sim N(0, \sigma_\nu^2)$ (see Dhyne *et al.* (2007) for details).

Dhyne *et al.* (2007) report results of estimated values of $\sigma_{\nu i}$ and S_i as well as the actual average price change $m_{1i} = E(\Delta p_t | p_t \neq p_{t-1}, p_t = ... = p_0)$ for a cross section of products indexed by i. We restrict our sample to products for which the proportion of price increases is greater than 70% (56 products), to get closer to the framework presented above with only price increases. We use results reported in their paper (Tables A and B of their appendix) to plot Figure 9 which is the superposition of two (cross-sectional) scatter plots: the average of observed price changes m_{1i} and the size of the band S_i, both as a function of the standard deviation of the band $\sigma_{\nu i}$.

[Figure 9 here]

First, we observe that for every product, m_{1i} is smaller than S_i. The difference between the two values can be very large, the median of m_{1i}'s is close to 5% whereas the median of estimated S_i's is closer to 40%. This result illustrates Property 1.2. Figure 9 also indicates that for higher standard deviations of the band $\sigma_{\nu i}$, the gap between the size of band S_i and the average price change m_{1i} increases. For large values of $\sigma_{\nu i}$, the estimated size of the band is clearly not informative about the observed average price change.

Based on our previous results, an interpretation for the gap observed in the rightmost part of Figure 9 emerges. Observations on the rightmost part of the figure are characterized by a large value of $\sigma_{\nu i}$, and therefore presumably associated with sectors with a high variance of price changes. Variability in the size of price adjustment is a well documented fact in micro price data. From our simulation exercise, we know that to fit a large variability in price changes m_{2i}, while the average size of price changes m_{1i} is the roughly the same for all products, the procedure needs to assume a large bandsize parameter S_i. Our interpretation is that a large level and variance of the band are here necessary to match the variance of price changes.

An alternative insight is obtained from the adjustment hazard function. On Figure 3, we have plotted the hazard function associated to $S = 40$ and $\sigma_S = 20$, which corresponds to the median values of the scatter plot of Figure 9. For these values, the hazard is nearly flat but non-zero near $t = 1$, so that small price changes are allowed. Consistent with our results of section 3 and 4, large estimates of σ_S and S reflect the prevalence of small price changes and for some items, the flatness of the hazard function (consistent with the predictions of the Calvo model).

Large estimates of S suggest that the average menu cost is sizeable. The structural model with stochastic adjustment costs used in Section 2 provides additional insights. Figure 10 plots the distribution of menu costs when prices are adjusted and the same distribution when prices are kept unchanged. These two distributions are noticeably different. The average menu cost at price changes is equal to 2.6%, whereas it is only 4.1% when there is no price change. On average, firms are more likely to adjust their prices when they face lower adjustment costs as pointed out by Willis (2000). As a result, the distribution of adjustment costs is not informative about patterns of adjustment costs actually observed when prices are changed.

[Figure 10 here]

6 Conclusion

We have illustrated some properties of (S, s) models with time-varying random thresholds, an empirical specification increasingly used to model data featuring infrequent adjustment. First, the adjustment hazard is shown to decrease with the size of the threshold and to flatten with the variance of the band. Second, this model is able to produce a large variety of hazard functions. Two polar cases are the constant hazard model (Calvo, 1983) (for large values of both S and σ_S) and the traditional fixed band (S, s) model (for $\sigma_S = 0$). Third, the average size of price changes is an increasing function of the mean band size, and is always smaller than this band. The average size of price changes decreases with the variance of the band. Finally, the variance of the size of adjustments is an increasing function of the variance of the band, and, for relatively low values of σ_S and large values of S, this moment is not sensitive to the mean of the band.

These results have important implications for the interpretation of stochastic (S, s) model estimates. An abundant empirical literature dealing with investment, durable consumption, hiring or price-setting decisions report a high variability of the size of adjustments at the microeconomic level. Estimating (S, s) models with time-varying random thresholds is an appealing solution in this context. As illustrated by our results, this flexible specification indeed allows to match data featuring infrequent microeconomic adjustments of variable size. Our results however show that to match a large variance of adjustments for a given mean of these adjustments, these models need to produce large estimates for bandsize parameters. Contrary to the core (S, s) model with fixed bands, the estimated size of the band is then not informative about the size of actual adjustments anymore. Large estimated average threshold (and implicitly large menu costs) do not imply the econometric rejection of the model. Those cases may however raise the question of whether substantial fluctuations in menu costs are an economically plausible mechanism.

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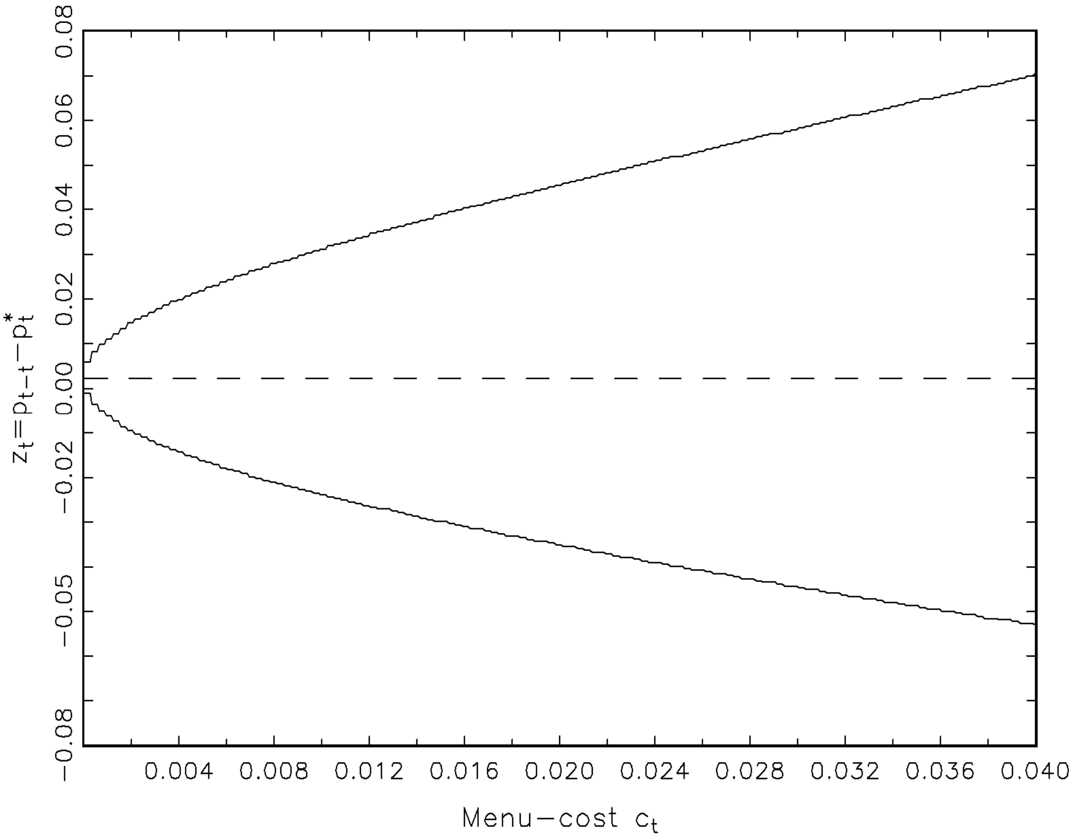
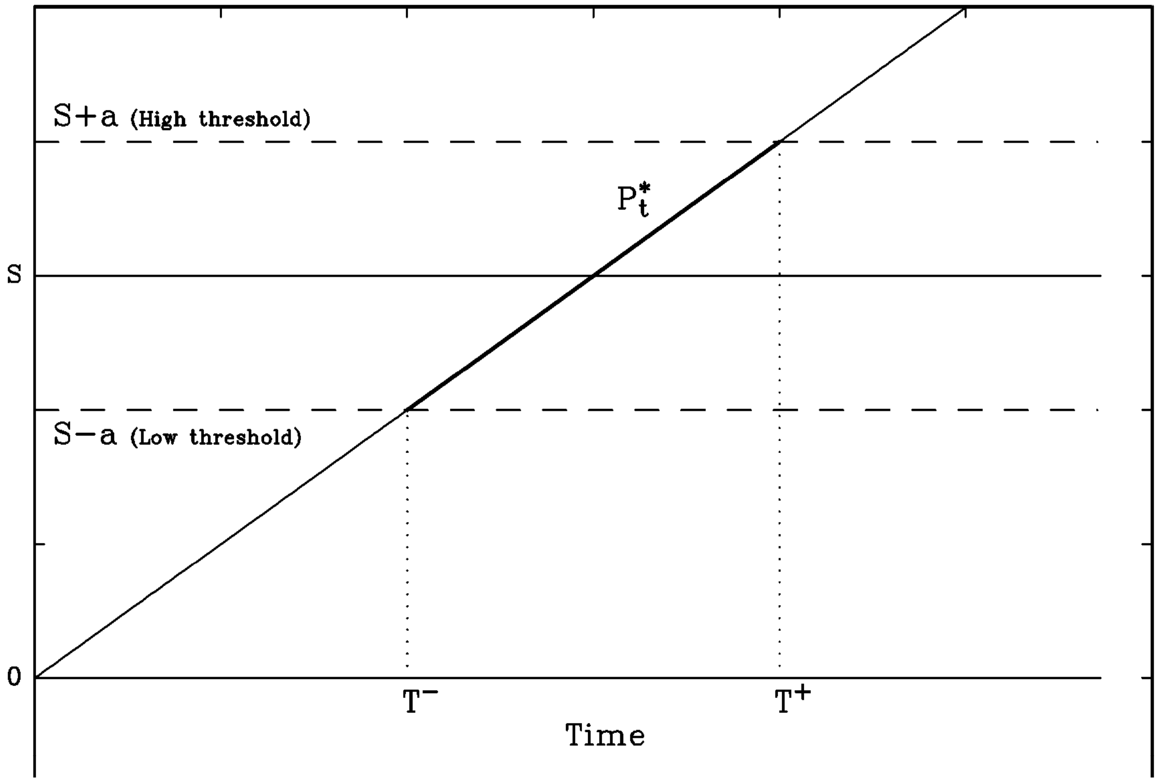
Figure 1: Decision rule derived from the structural random menu cost model



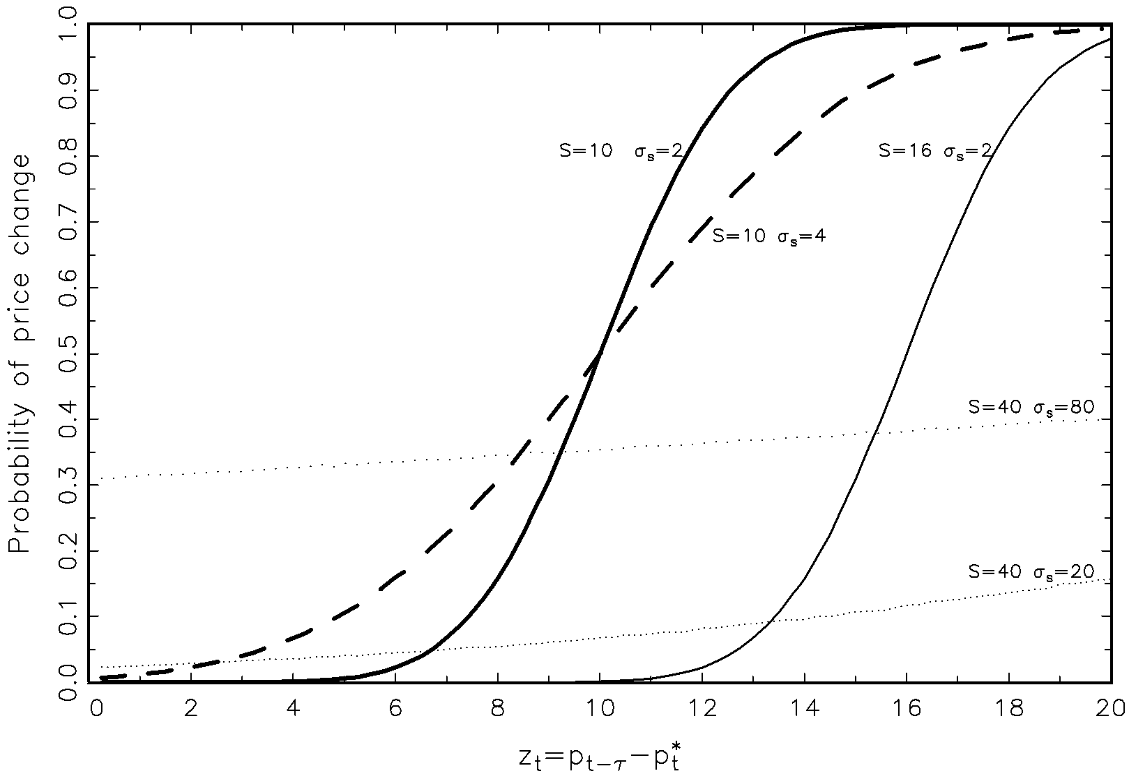
Note: x-axis: realizations of the menu cost; y-axis: values of the price gap $z_t = p_{t-\tau} - p_t^*$, where τ is the duration elapsed since the last price change. As long as the price gap remains inside the region delimited by the solid line (region of inaction), the firm does not change its price. Outside this region, the price is adjusted. When the price is adjusted, the price gap z_t comes back to the reset point (dotted line).

Figure 2: Trajectory of the price deviation in our simple model



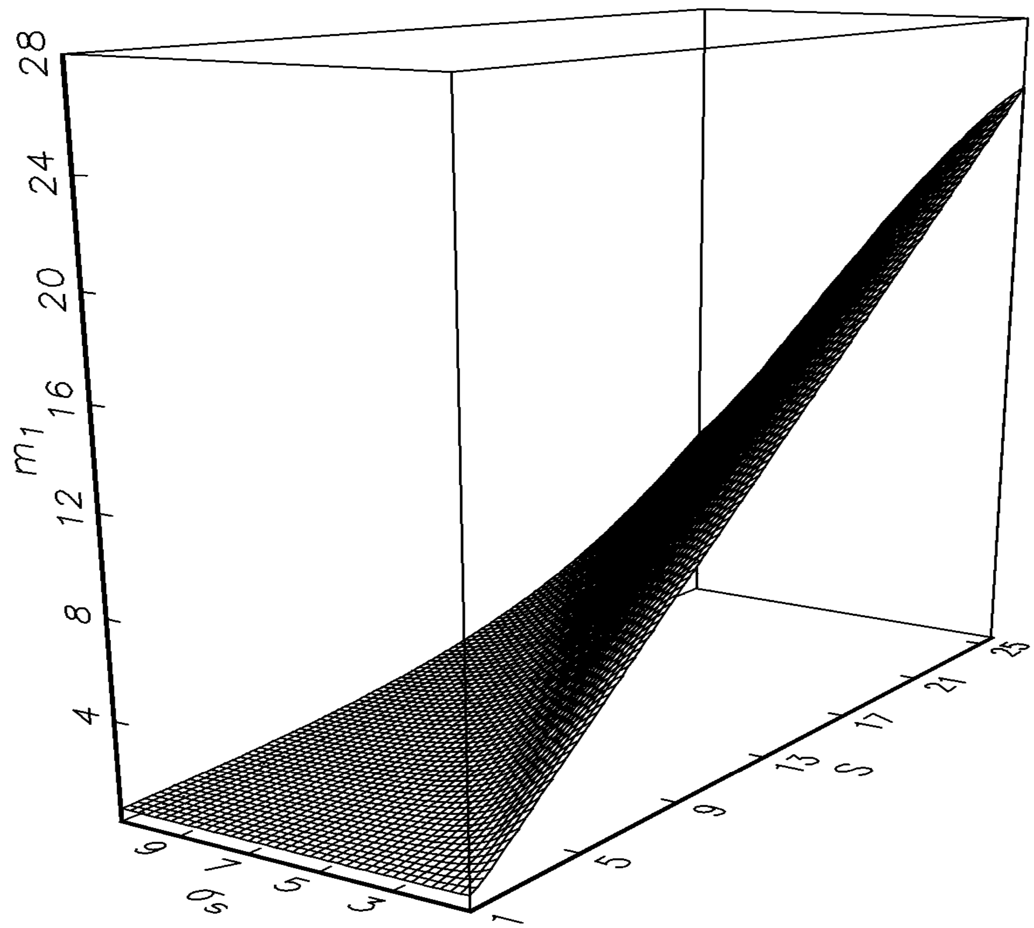
Note: x-axis: time; y-axis: values of the price gap $z_t = p_t^* = \gamma t$. The price deviation p_t^* starts at time 0 and grows until it hits the random threshold S_t. This threshold can take two values $S - a$ and $S + a$ with probability $\frac{1}{2}$. The price can only be adjusted between $t = T^-$ and $t = T^+$ (bold line).

Figure 3: Adjustment hazard as a function of model parameters (S, σ_S)



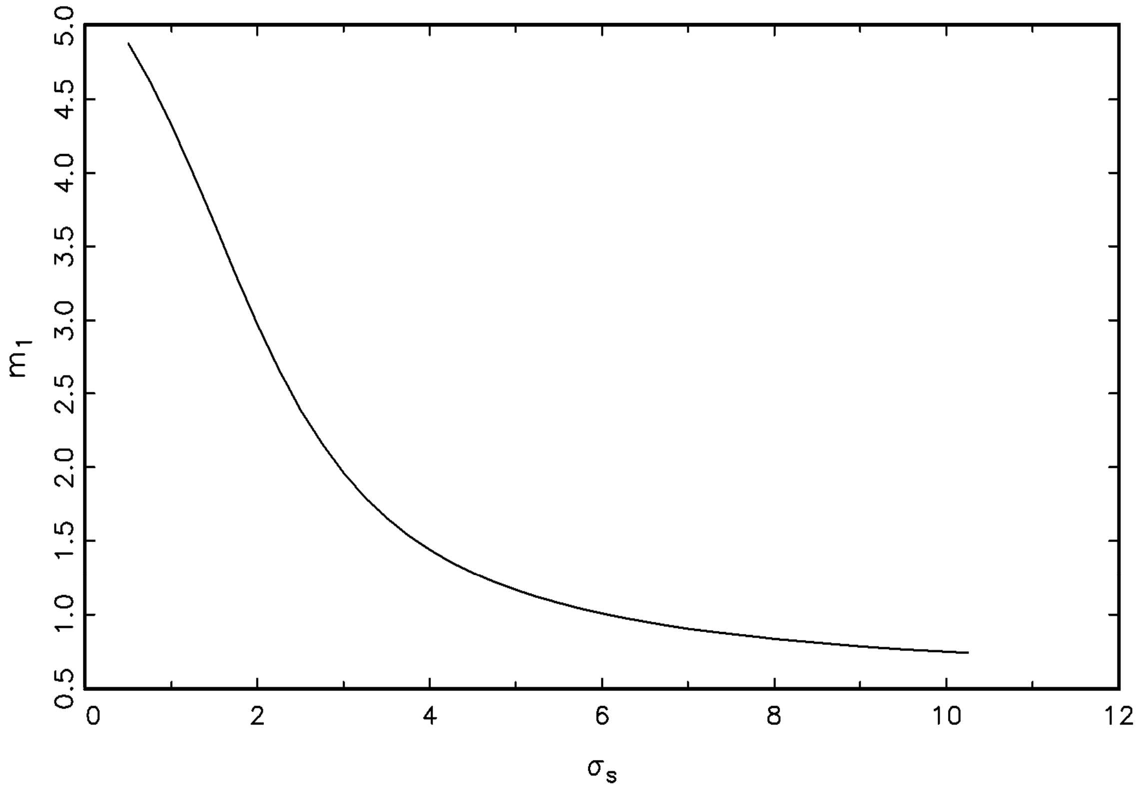
Note: x-axis: values of the price gap $z_t = p_{t-\tau} - p_t^*$, where τ is the duration elapsed since the last price change. y-axis: the probability of price change. Each line is generated for different values of S and σ_S.

Figure 4: The average size of price changes as a function of model parameters (S, σ_S)



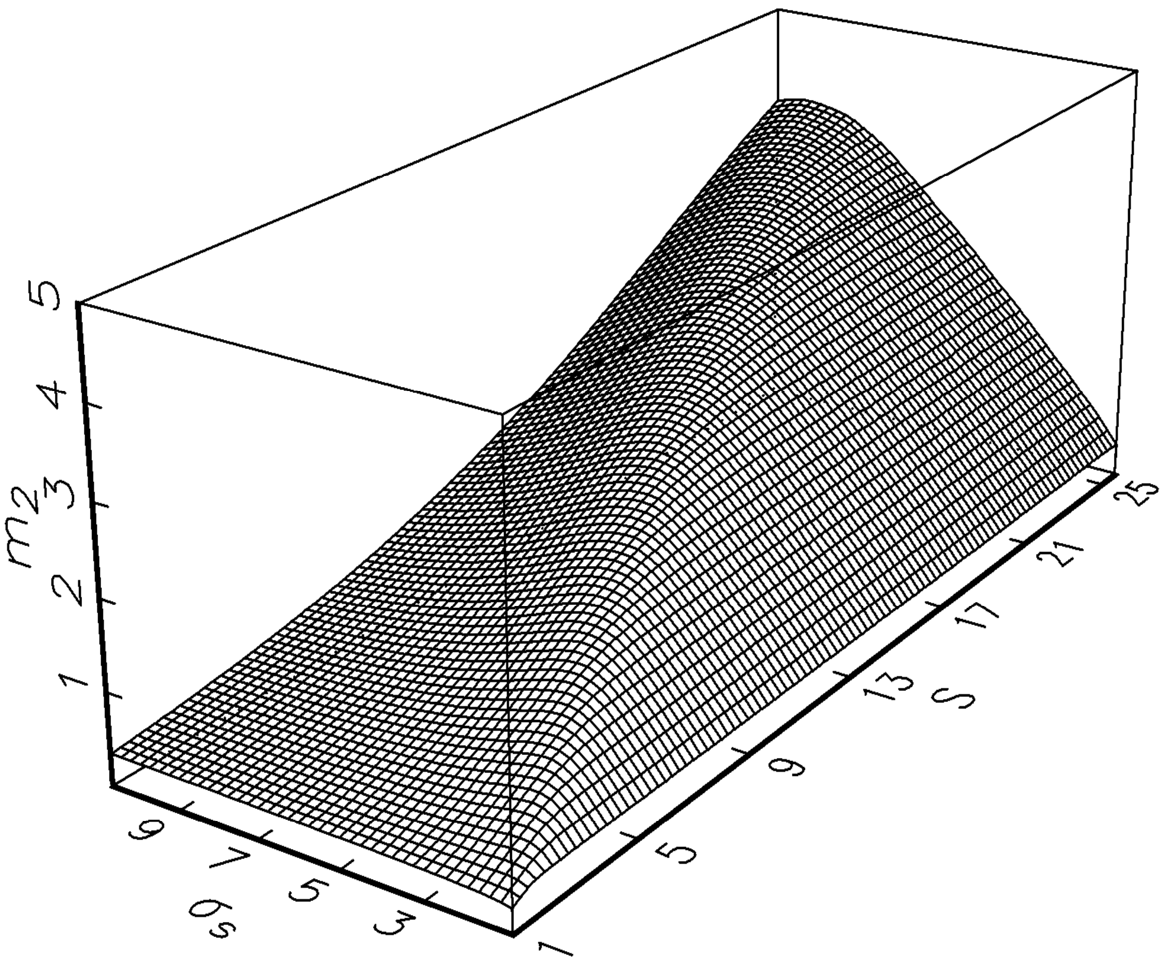
Note: x-axis: σ_S, the standard deviation of S_t y-axis: S, the mean of S_t and z-axis: m_1, the average size of price changes.

Figure 5: The average size of price changes m_1 as a function of the variance of the band σ_S ($S = 5$)



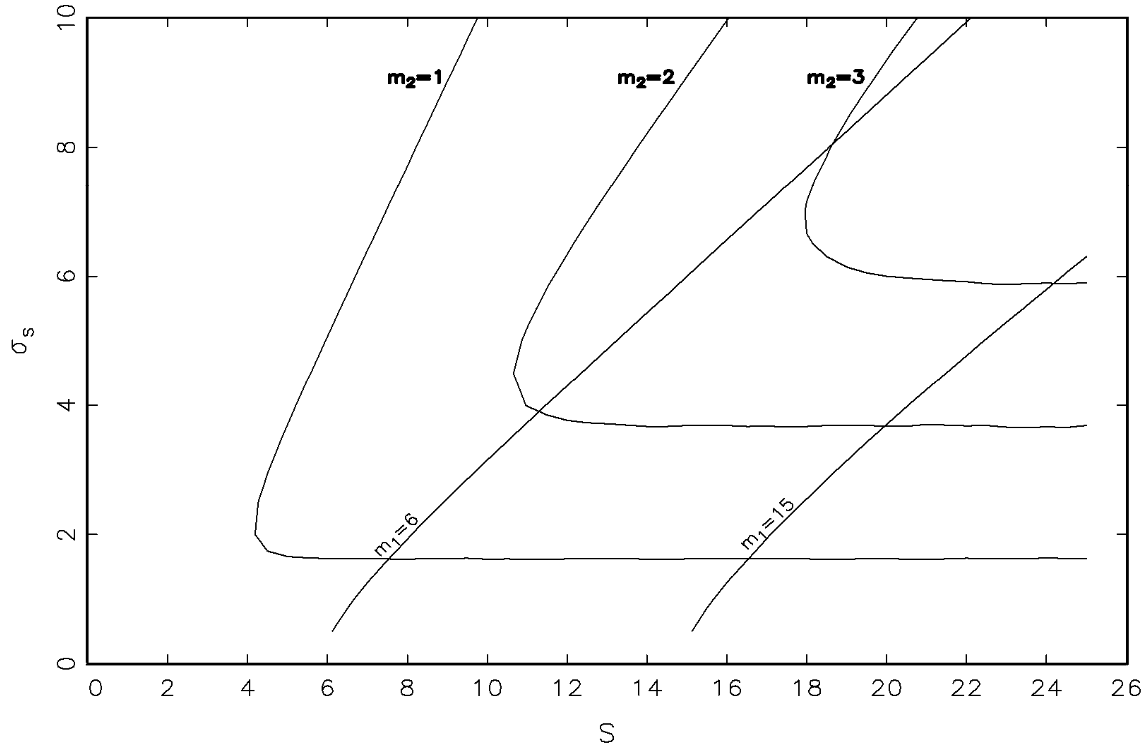
Note: x-axis: σ_S the standard deviation of S_t y-axis: m_1 the average size of price changes.

Figure 6: Standard deviation of price changes as a function of model parameters (S, σ_S)



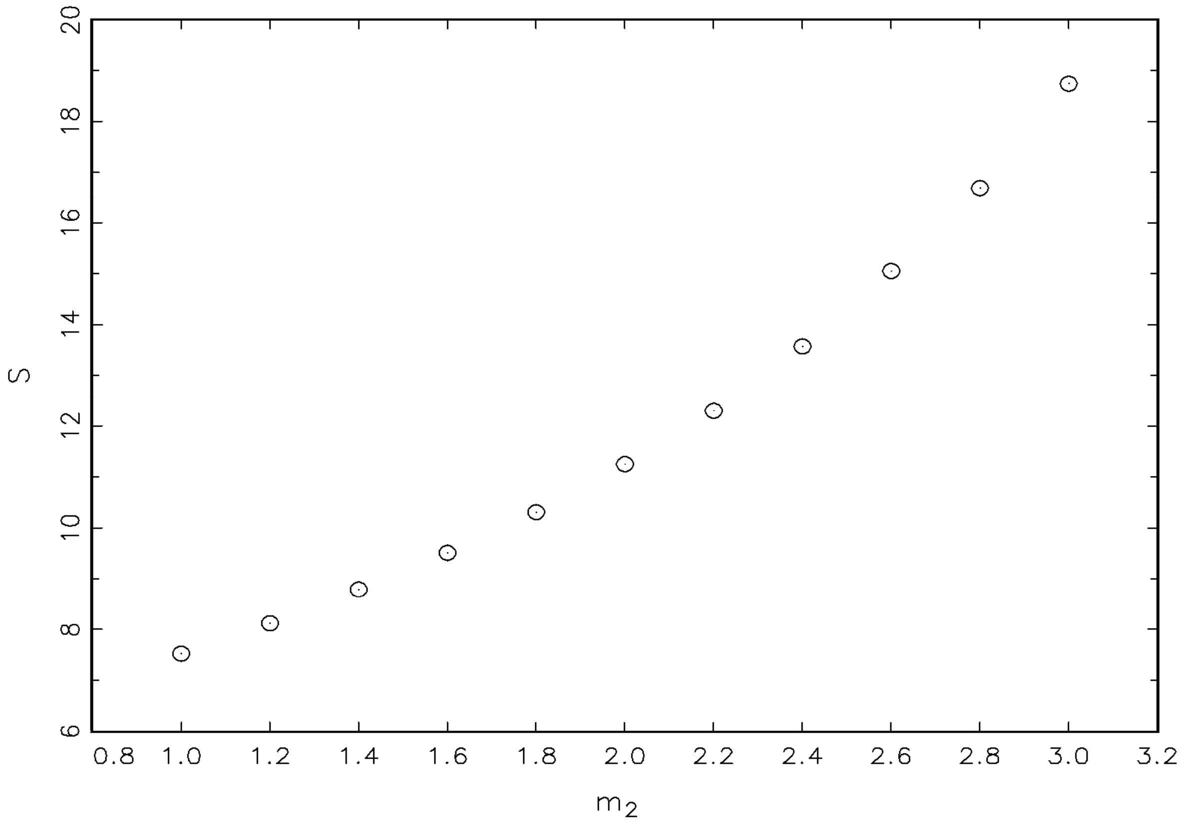
Note: x-axis: σ_S the standard deviation of S_t y-axis: S the mean of S_t and z-axis: m_2 the standard deviation of the size of price changes.

Figure 7: Contour plots of mean and variance of price changes in the plane (S, σ_S)



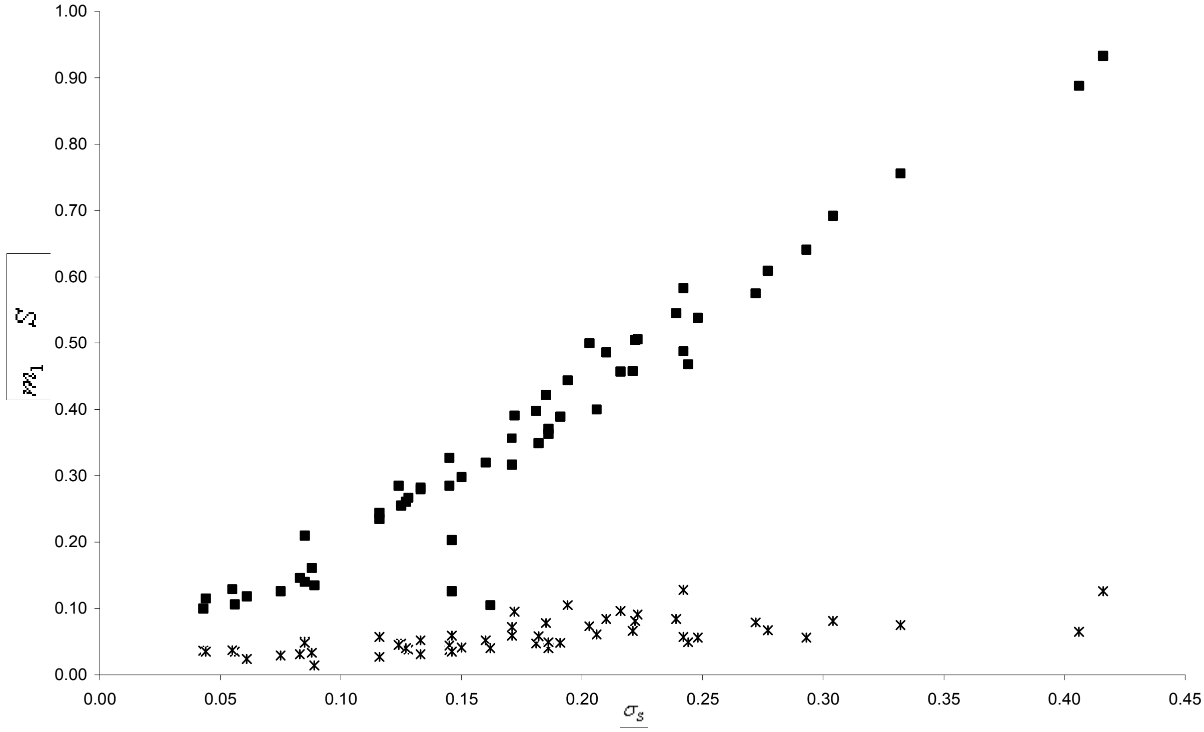
Note: x-axis: S the mean of S_t and y-axis: σ_S the standard deviation of S_t. Each line is a contour plot for different values of m_1 and m_2 where m_1 is the average size of price changes and m_2 is the standard deviation of the size of price changes.

Figure 8: Values of S matching moments ($m_1 = 6$ and $m_2 \in [1;3]$) - Minimum distance estimates



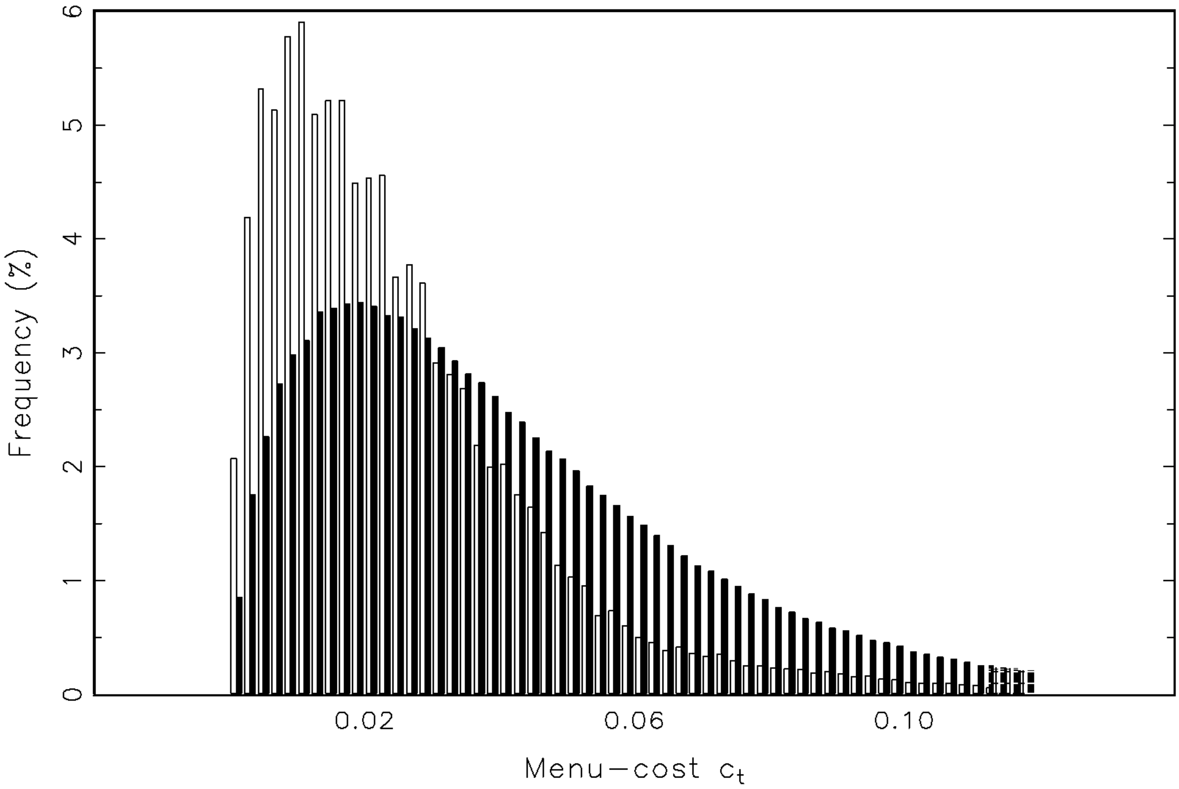
Note: x-axis: m_2 the standard deviation of the size of price changes and y-axis: S the average of S_t, which allows to fit moments $m_1$and m_2.

Figure 9: Average price changes and S as a fuction of the variance of the band (source: Dhyne *et al.* (2007))



Note: for each sector, the average price change (crosses - y-axis) and the average size of the band (black squares - y-axis) are plotted against the residual variance of the band (x-axis).

Figure 10: Distribution of menu costs when price change and when prices are kept unchanged



Note: white bars: distribution of menu costs at price changes and black bars: menu costs when prices remain unchanged. Those distributions are simulated from the structural model of price setting with time-varying random menu costs presented in section 2 of the paper.

8 Appendix 1

Proposition 1

$$
\begin{aligned}
m_1(S,a,\gamma) &= S-a-\gamma+\gamma\left[\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[\frac{1}{2}\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^{k-1} + \left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[\frac{4}{2}\left(1-\left(\frac{a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right) + \left(\frac{2a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[2-2\left(\frac{a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} + 2\left(\frac{a}{\gamma}+\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a-\gamma+\gamma\left[2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right] \\
&= S-a+\gamma\left[1-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right]
\end{aligned}
$$

In the above we have used the following property.

We define $G(x) = \sum_{j=0}^{J} x^j = \frac{1-x^{J+1}}{1-x}$

Then: $G'(x) = \sum_{j=0}^{J} jx^{j-1} = \frac{(x-1)(J+1)x^J+1-x^{J+1}}{(1-x)^2}$

We have $x = \frac{1}{2}$ then

$$
\begin{aligned}
G'\left(\frac{1}{2}\right) &= \frac{-(J+1)\left(\frac{1}{2}\right)^{J+1}+1-\left(\frac{1}{2}\right)^{J+1}}{\left(1-\frac{1}{2}\right)^2} \\
&= 4\left[\left(\frac{1}{2}\right)^{J+1}(-1-J-1)+1\right] \\
&= 4-\left(\frac{1}{2}\right)^{J-1}(J+2)
\end{aligned}
$$

If $J = \frac{2a}{\gamma}$

$$G'\left(\frac{1}{2}\right) = 4 \times \left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} \left(\frac{a}{\gamma} + 1\right)\right)$$

Property 1.2

Let us define $f(a) = \gamma \left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right) - a$

Then,

$$\begin{aligned}
\frac{\partial f}{\partial a} &= 2\ln 2 \times \exp\left[-\frac{2\ln 2}{\gamma} a\right] - 1 \\
\frac{\partial^2 f}{\partial a^2} &= -\frac{(2\ln 2)^2}{\gamma} \times \exp\left[-\frac{2\ln 2}{\gamma} a\right] < 0
\end{aligned}$$

We have also $\frac{\partial f}{\partial a}\left(\frac{\gamma}{2}\right) = \ln 2 - 1 < 0$ and $\lim_{\infty} \frac{\partial f}{\partial a}(a) = -1$.

Then, f is a decreasing function of a. We also find that $f(\frac{\gamma}{2}) = \gamma\left(1 - \frac{1}{2}\right) - \frac{\gamma}{2} = 0$. We can then conclude that $m_1(S, a, \gamma) < S$ for all values of a different from $\frac{\gamma}{2}$. In the specific case $a = \frac{\gamma}{2}$, $m_1(S, a, \gamma) = S$, because of a degenerate distribution of price changes. In that case, the step of the drift in the optimal price (γ) is exactly equal to the distance between the upper and lower value of the band ($2a$), and price changes can only take two values $S - a$ and $S + a$ with probability $\frac{1}{2}$.

Property 1.3

$$\begin{aligned}
m_1(S, a, \gamma) &= S - a + \gamma\left(1 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right) \\
&= S - a + \gamma\left(1 - \exp\left[-\frac{2\ln 2}{\gamma} a\right]\right)
\end{aligned}$$

Then,

$$\frac{\partial m_1(S, a, \gamma)}{\partial a} = 2\ln 2 \times \exp\left[-\frac{2\ln 2}{\gamma} a\right] - 1$$

Let define $g(a) = 2\ln 2 \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] - 1$

$$g'(a) = -\frac{(2\ln 2)^2}{\gamma} \times \exp\left[-\frac{2\ln 2}{\gamma}a\right] < 0$$

We have also $g\left(\frac{\gamma}{2}\right) = \ln 2 - 1 < 0$ and $\lim_{\infty} g(a) = -1$.

The main result is then the following:

$$\frac{\partial m_1(S,a,\gamma)}{\partial a} < 0 \quad \text{for all } a \geq \frac{\gamma}{2}$$

Proposition 2

$$m_2(S,a,\gamma) = \gamma^2 \left[\sum_{k=1}^{\frac{2a}{\gamma}} k^2 \left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)^2 \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2 - \left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \right]$$

where $\left(E\left(\widetilde{\tau}\right)\right)^2 = \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2$

$$
\begin{aligned}
m_2(S,a,\gamma) &= \gamma^2\left[\sum_{k=1}^{\frac{2a}{\gamma}} k^2\left(\frac{1}{2}\right)^k + \left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right] \\
&= \gamma^2\left[\frac{1}{4}\sum_{k=1}^{\frac{2a}{\gamma}} k(k-1)\left(\frac{1}{2}\right)^{k-2} + \frac{1}{2}\sum_{k=1}^{\frac{2a}{\gamma}} k\left(\frac{1}{2}\right)^{k-1} + \left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2\right] \\
&= \gamma^2\left[\begin{array}{r} \frac{1}{4}\left(16-\left(\left(\frac{2a}{\gamma}\right)^2+\frac{6a}{\gamma}+4\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}-2}\right)+2\left(\frac{2a}{\gamma}\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}+1}-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\left(\frac{2a}{\gamma}+1\right)+1\right) \\ +\left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}-\left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \end{array}\right] \\
&= \gamma^2\left[\begin{array}{r} 4-\left(\left(\frac{2a}{\gamma}\right)^2+\frac{6a}{\gamma}+4\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}+\frac{2a}{\gamma}\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\left(\frac{4a}{\gamma}+2\right)+2 \\ +\left(\frac{2a}{\gamma}+1\right)^2\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}-\left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \end{array}\right] \\
&= \gamma^2\left[\begin{array}{r} 4+2+\left(-\left(\frac{2a}{\gamma}\right)^2+\left(\frac{2a}{\gamma}\right)^2-\frac{6a}{\gamma}+\frac{2a}{\gamma}-\frac{4a}{\gamma}+\frac{4a}{\gamma}-4-2+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} \\ -\left(2-\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}\right)^2 \end{array}\right] \\
&= \gamma^2\left[6+\left(-\frac{4a}{\gamma}-5\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}-4+4\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}-\left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right] \\
&= \gamma^2\left[2-\left(\frac{4a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}}-\left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right]
\end{aligned}
$$

In the above we have used the following property.

We find $G'(x) = \sum_{j=0}^{J} jx^{j-1} = \frac{(x-1)(J+1)x^J+1-x^{J+1}}{(1-x)^2}$

Then: $G''(x) = \sum_{j=0}^{J} j(j-1)x^{j-2} = \frac{2G'(x)-J(J+1)x^{J-1}}{1-x}$

We have $x = \frac{1}{2}$ then

$$\begin{aligned}
G''\left(\frac{1}{2}\right) &= \frac{2G'\left(\frac{1}{2}\right) - J(J+1)\left(\frac{1}{2}\right)^{J-1}}{\frac{1}{2}} \\
&= 4G'\left(\frac{1}{2}\right) - 2J(J+1)\left(\frac{1}{2}\right)^{J} \\
&= 16 - 4\left(\frac{1}{2}\right)^{J-1}(J+2) - 2J(J+1)\left(\frac{1}{2}\right)^{J-1} \\
&= 16 - (J^2+3J+4)\left(\frac{1}{2}\right)^{J-2}
\end{aligned}$$

If $J = \frac{2a}{\gamma}$

$$G''\left(\frac{1}{2}\right) = 16 - \left[\left(\frac{2a}{\gamma}\right)^2 + \frac{6a}{\gamma} + 4\right]\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}-2}$$

Property 2.2

$$\begin{aligned}
m_2(S,a,\gamma) &= \gamma^2\left[2 - \left(\frac{4a}{\gamma}+1\right)\left(\frac{1}{2}\right)^{\frac{2a}{\gamma}} - \left(\frac{1}{2}\right)^{\frac{4a}{\gamma}}\right] \\
&= \gamma^2\left[2 - \left(\frac{4a}{\gamma}+1\right)\exp\left(-\frac{2\ln 2}{\gamma}a\right) - \exp\left(-\frac{4\ln 2}{\gamma}a\right)\right]
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial m_2(S,a,\gamma)}{\partial a} &= \gamma^2\left[\begin{array}{c} -\frac{2\ln(2)}{\gamma}\left(\frac{4a}{\gamma}+1\right)\exp\left(-\frac{2\ln 2}{\gamma}a\right) + \frac{4}{\gamma}\exp\left(-\frac{2\ln 2}{\gamma}a\right) \\ +\frac{4\ln(2)}{\gamma}\exp\left(-\frac{4\ln 2}{\gamma}a\right) \end{array}\right] \\
&= 2\gamma\exp\left(-\frac{2\ln 2}{\gamma}a\right)\left[-\frac{4a\ln(2)}{\gamma} - \ln(2) + 2 + 2\ln(2)\exp\left(-\frac{2\ln 2}{\gamma}a\right)\right]
\end{aligned}$$

Let define $h(a) = -\frac{4a\ln(2)}{\gamma} - \ln(2) + 2 + 2\ln(2)\exp\left(-\frac{2\ln 2}{\gamma}a\right)$

$$\frac{\partial h(a)}{\partial a} = -\frac{4\ln(2)}{\gamma} - \frac{(2\ln 2)^2}{\gamma}\exp\left(-\frac{2\ln 2}{\gamma}a\right) < 0$$

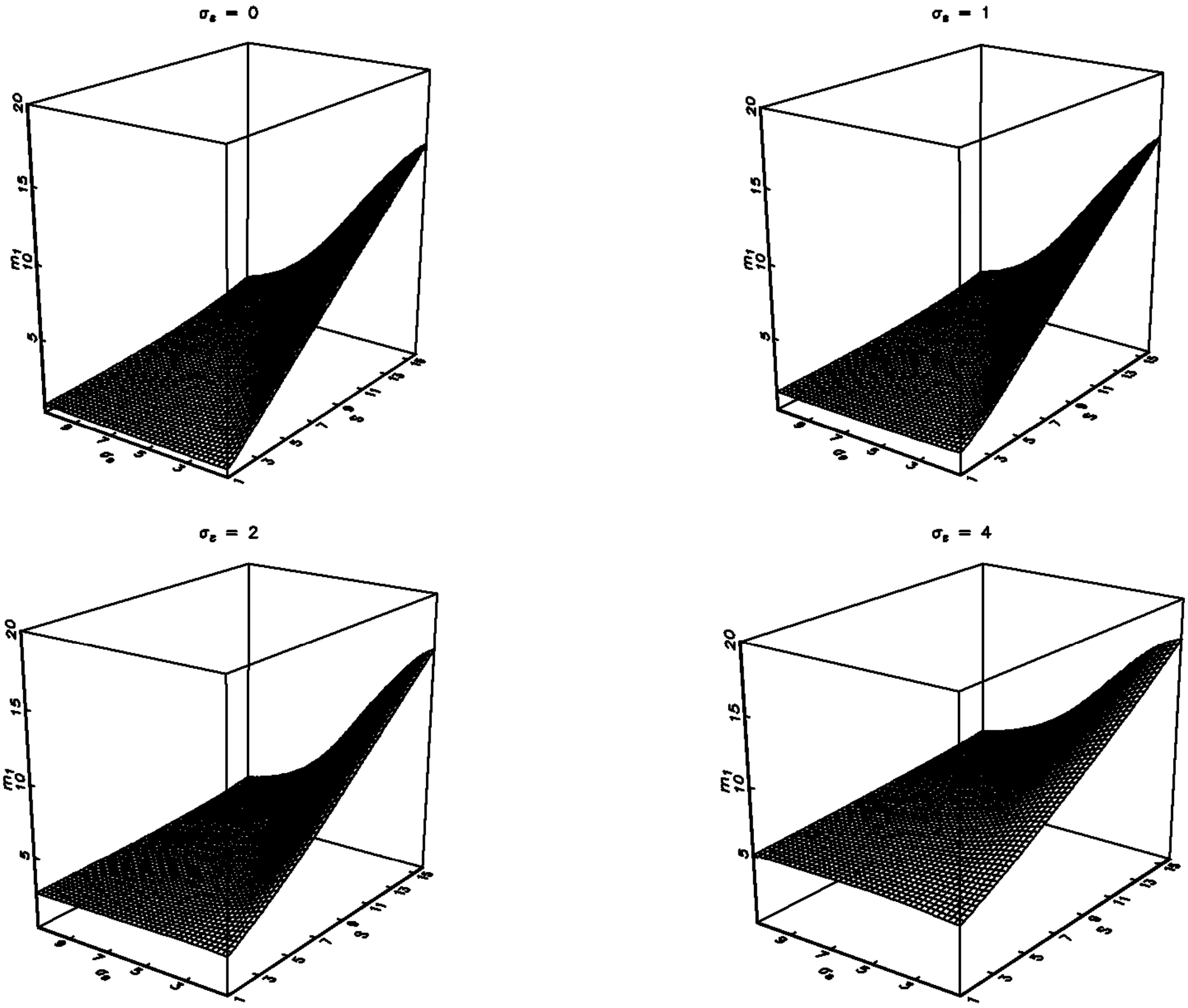
$h\left(\frac{\gamma}{2}\right) = 2(1-\ln(2)) > 0$ and $\lim_{\infty} h(a) = 0$

The main result is then the following:

$$\frac{\partial m_2(S, a, \gamma)}{\partial a} > 0 \quad \text{for all } a \geq \frac{\gamma}{2}$$

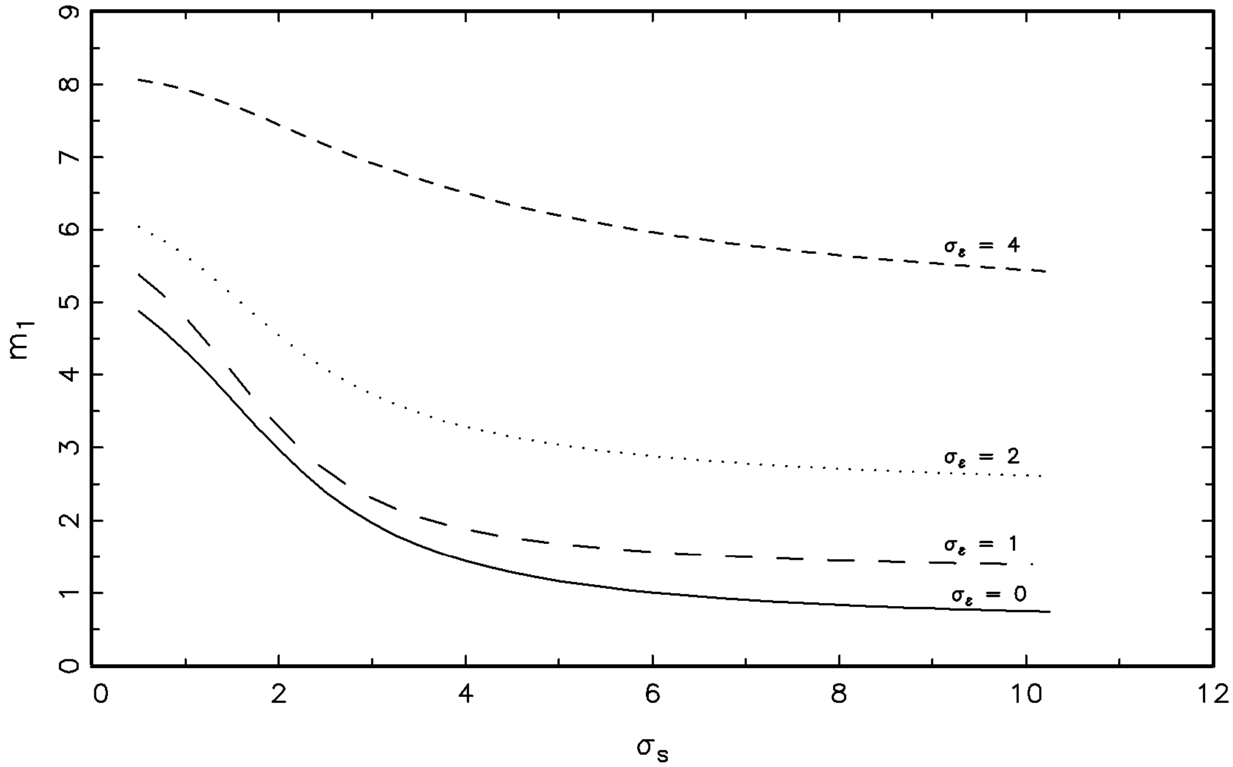
9 Appendix 2

Figure A: The average size of price changes as a function of model parameters (S, σ_S) for different values of σ_ε



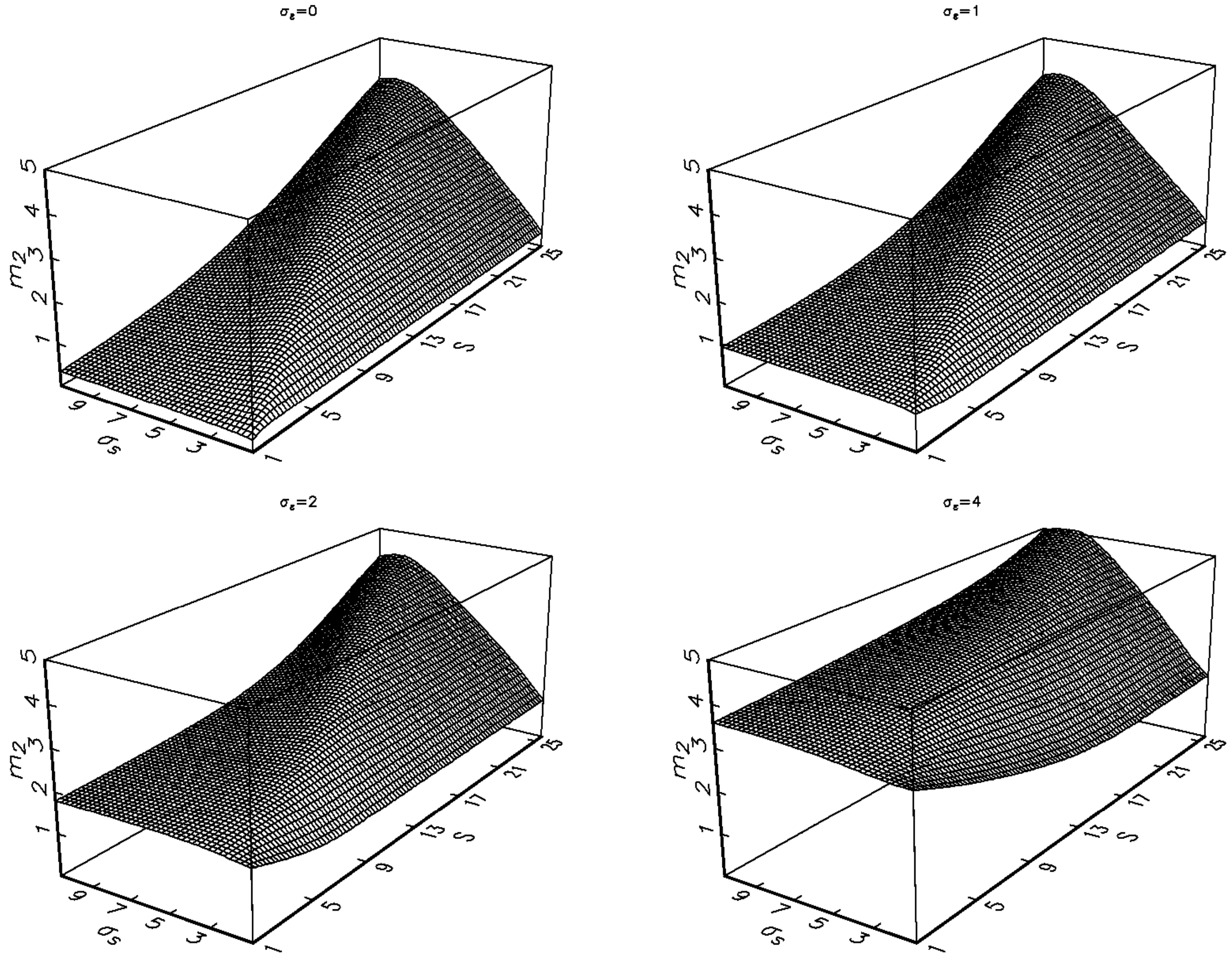
Note: x-axis: σ_S the standard deviation of S_t y-axis: S the mean of S_t and z-axis: m_1 the average size of price changes. Each of the four graphs corresponds to a given value of σ_ε, the standard deviation of the idiosyncratic shock when $p_t^* = \gamma t + \varepsilon_t$.

Figure B: The average size of price changes m_1 as a function of the variance of the band σ_S ($S = 5$) for different values of σ_ε



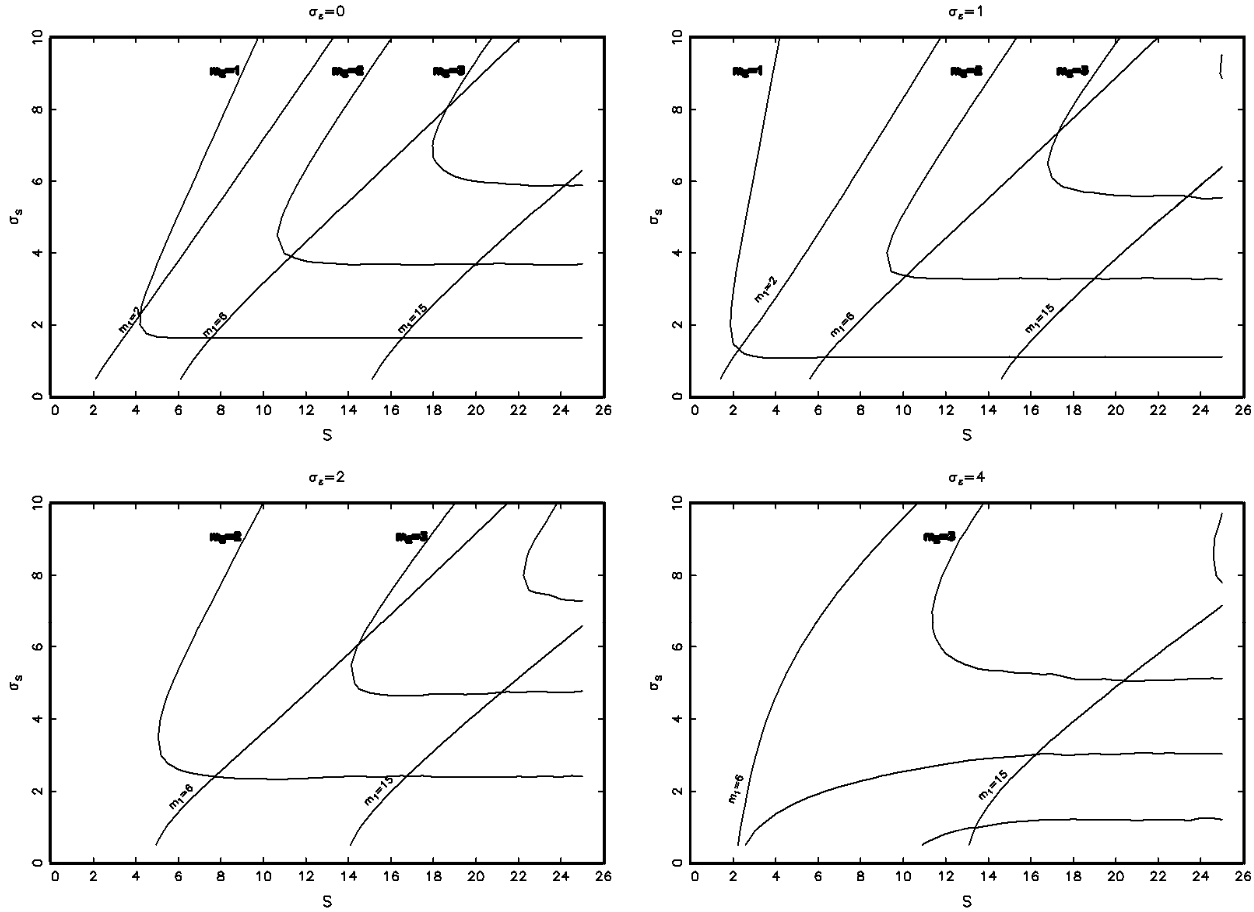
Note: x-axis: σ_S the standard deviation of S_t y-axis: m_1 the average size of price changes. Each of the four lines corresponds to a given value of σ_ε, the standard deviation of the idiosyncratic shock when $p_t^* = \gamma t + \varepsilon_t$.

Figure C: Standard deviation of price changes as a function of model parameters (S, σ_S) for different values of σ_ε



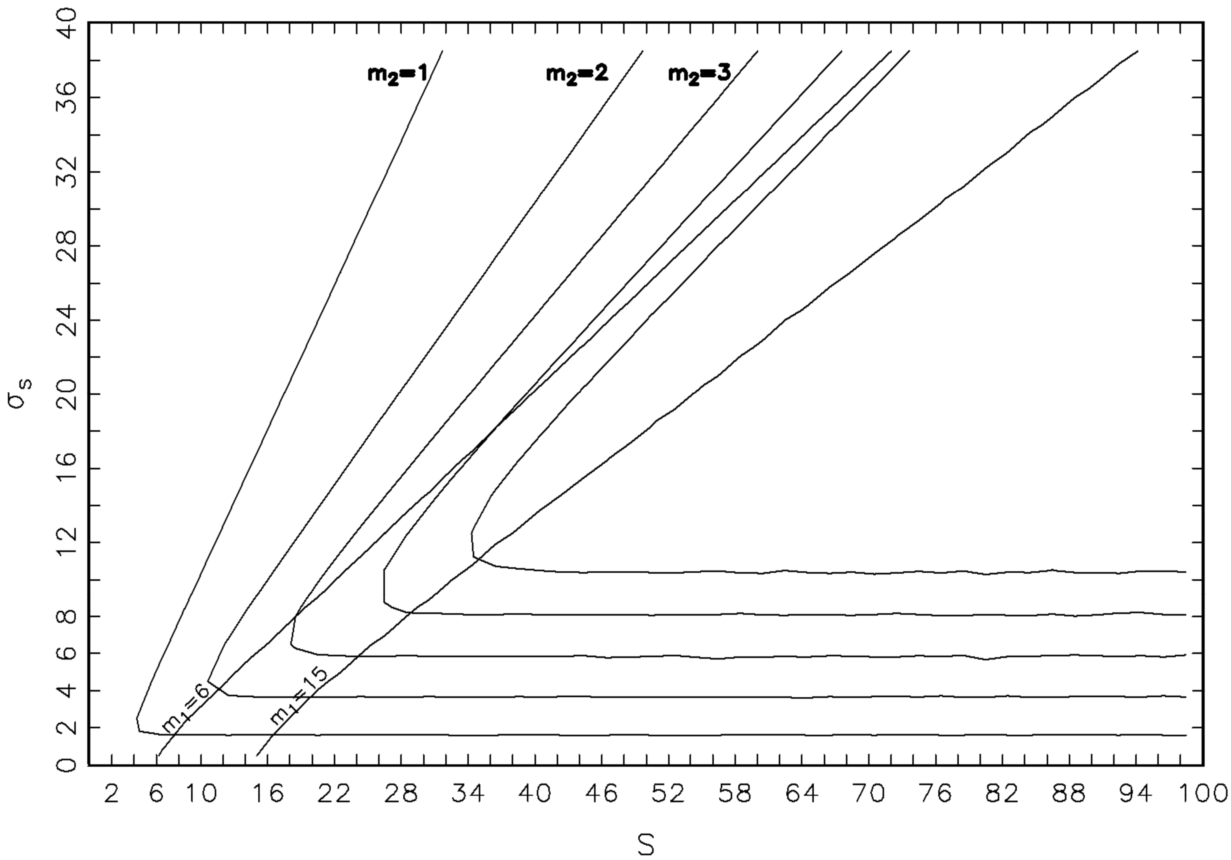
Note: x-axis: σ_S the standard deviation of S_t y-axis: S the mean of S_t and z-axis: m_2 the standard deviation of the size of price changes. Each of the four graphs corresponds to a given value of σ_ε, the standard deviation of the idiosyncratic shock when $p_t^* = \gamma t + \varepsilon_t$.

Figure D: Contour plots of mean and variance of price changes in the plane (S, σ_S) for different values of σ_ε



Note: x-axis: S the mean of S_t and y-axis: σ_S the standard deviation of S_t. Each line is a contour plot for different values of m_1 and m_2 where m_1 is the average size of price changes and m_2 is the standard deviation of the size of price changes. Each of the four graphs corresponds to a given value of σ_ε, the standard deviation of the idiosyncratic shock when $p_t^* = \gamma t + \varepsilon_t$.

Figure E: Contour plots of mean and variance of price changes in the plane (S, σ_S) for large values of S and σ_S



Note: x-axis: S the mean of S_t and y-axis: σ_S the standard deviation of S_t. Each line is a contour plot for different values of m_1 and m_2 where m_1 is the average size of price changes and m_2 is the standard deviation of the size of price changes.